Fast and simple constant-time hashing to the BLS12-381 elliptic curve

(and other curves, too!)

Riad S. Wahby, Dan Boneh

Stanford

August 26^{th}, 2019

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- Also: VRFs,

fixed-modulus arithmetic only

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- Widely used curve for $\approx$120-bit security level

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Why the BLS12-381 pairing-friendly elliptic curve?

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  - ☞ ZK proofs, signatures, IBE, ABE, . . .

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   - ☞ Open-source impls in C, Rust, Python, ...

Roadmap

1. Hash functions to elliptic curves

2. Optimizing the map of [BCIMRT10]

3. Evaluation results

Notation

$\mathbb{F}_p$ is the finite field of integers mod a prime p

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$\mathbb{G} \subseteq E(\mathbb{F}_p)$ is a subgroup of prime order q.
$\#E(\mathbb{F}_p) = hq$; h is the *cofactor*.

Hash and check

$\text{HashToCurve}_{\text{H\&C}}(\text{msg})$:

$\text{ctr} \leftarrow 0$

$y \leftarrow \perp$

while $y = \perp$:

$\quad x \leftarrow H_p(\text{ctr} \,||\, \text{msg})$

$\quad \text{ctr} \leftarrow \text{ctr} + 1$

$\quad ySq \leftarrow x^3 + ax + b$

$\quad y \leftarrow \text{sqrt}(ySq)$ // $\perp$ if ySq is non-square

$P \leftarrow (x, y)$

return P^h // map to $\mathbb{G}$ via cofactor mul

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HashToCurve$_{\mathsf{H\&C}}$(msg):

```
ctr ← 0
y ← ⊥
while y = ⊥:
    x ← H_p(ctr || msg)
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✗ Loop a fixed number of times?

Slow; well-meaning "optimization" breaks CT.

Deterministic maps to elliptic curves

$M : \mathbb{F}_p \to E(\mathbb{F}_p)$, where $E : y^2 = x^3 + ax + b$ and $p > 5$:

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Hash functions from deterministic maps

Compose H_p and M in a natural way:

$\mathsf{HashToCurve}_{\mathsf{NU}}(\mathsf{msg})$:

$t \leftarrow H_p(\mathsf{msg})$ // $\{0,1\}^\star \to \mathbb{F}_p$

$P \leftarrow M(t)$ // $\mathbb{F}_p \to E(\mathbb{F}_p)$

return P^h // $E(\mathbb{F}_p) \to \mathbb{G}$

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This *could* be OK—but what if we need uniformity?

Uniform hashing from deterministic maps

For uniformity [BCIMRT10,FFSTV13]:

$$\begin{aligned}
&\mathsf{HashToCurve}(\mathsf{msg}): \\
&\quad P_1 \leftarrow M(H_p(0 \,||\, \mathsf{msg})) \\
&\quad P_2 \leftarrow M(H_p(1 \,||\, \mathsf{msg})) \\
&\quad P \leftarrow P_1 \cdot P_2 \\
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☞ HashToCurve is *indifferentiable* from RO [MRH05]

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The Simplified SWU map [BCIMRT10]

$$E : y^2 = f(x) = x^3 + ax + b,\ ab \neq 0.$$

Idea: pick x s.t. $f(ux) = u^3 f(x)$.

☞ For u non-square $\in \mathbb{F}_p$, $f(x)$ or $f(ux)$ is square.

The Simplified SWU map [BCIMRT10]

$$E : y^2 = f(x) = x^3 + ax + b,\ ab \neq 0.$$

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☞ If $p \equiv 3 \bmod 4$, $u = -t^2$ is non-square, so:

$$X_0(t) \triangleq -\frac{b}{a}\left(1 + \frac{1}{t^4 - t^2}\right) \qquad X_1(t) \triangleq -t^2 X_0(t)$$

Evaluating the S-SWU map

$$\text{S-SWU}(t) \triangleq \begin{cases} (X_0(t), \sqrt{f(X_0(t))}) & \text{if } f(X_0(t)) \text{ is square} \\ (X_1(t), \sqrt{f(X_1(t))}) & \text{otherwise} \end{cases}$$

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Requires two exponentiations! Can we do better?

Eliminating an exponentiation

Recall: $f(x_1) = -t^6 f(x_0)$. So:

$$f(x_1)^{\frac{p+1}{4}} = \left(-t^6 f(x_0)\right)^{\frac{p+1}{4}}$$

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Legendre symbol!

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✓ $f(x_0)^{\frac{p+1}{4}}$ is $\sqrt{-f(x_0)}$ when $f(x_0)$ is non-square!

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✓ Straightforward to generalize to $p \equiv 1 \bmod 4$.

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Issue: S-SWU still does not work with $ab = 0$.

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Then: S-SWU to $E'(\mathbb{F}_p)$, isogeny map to $E(\mathbb{F}_p)$.

✓ Preserves well-distributedness of S-SWU.

Roadmap

1. Hash functions to elliptic curves

2. Optimizing the map of [BCIMRT10]

3. Evaluation results

Implementation, baselines, setup, method

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For $\mathbb{G}_1$ and $\mathbb{G}_2$, we implement:

Maps: hash-and-check; [SW06]; this work

Styles: full bigint; field ops only, non-CT and CT

Hashes: non-uniform; uniform

In total: 34 hash variants, 3520 lines of C.

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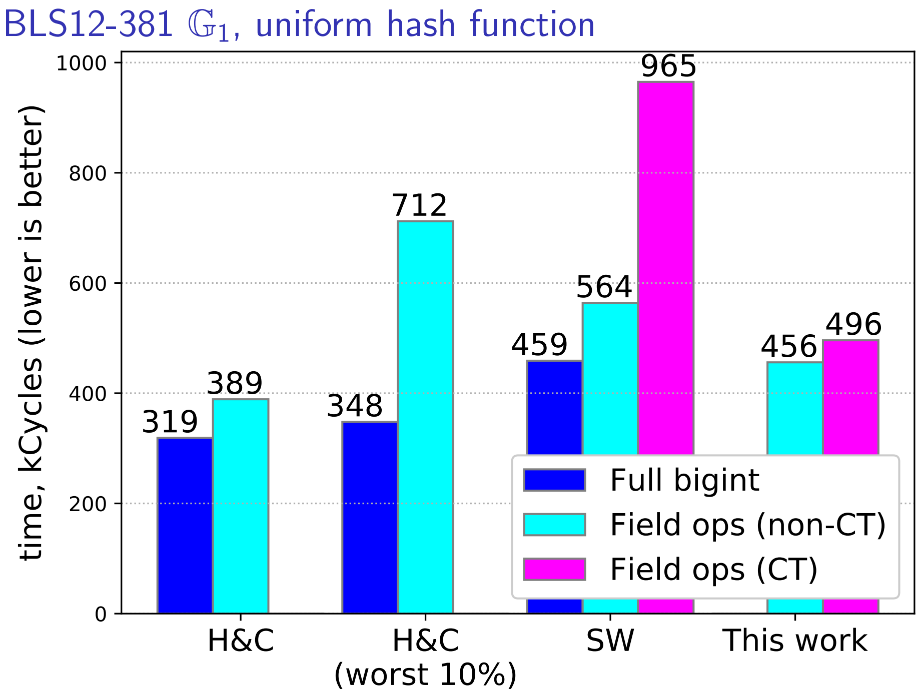
Method: run each hash 10^6 times; record #cycles.


BLS12-381 $\mathbb{G}_1$, uniform hash function
time, kCycles (lower is better)
0
200
400
600
800
1000
319
389
348
712
459
564
965
456
496
H&C
H&C (worst 10%)
SW
This work
Full bigint
Field ops (non-CT)
Field ops (CT)

Recap and conclusion

Contributions:

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`https://github.com/kwantam/bls12-381_hash`

`https://github.com/kwantam/bls_sigs_ref`

`rsw@cs.stanford.edu`