Classification of All t-Resilient Boolean Functions with $t + 4$ Variables

CASA
Cyber Security in the Age of Large-Scale Adversaries

FSE 2024, Leuven, Belgium, March 29, 2024.

Shahram Rasoolzadeh
Ruhr University Bochum

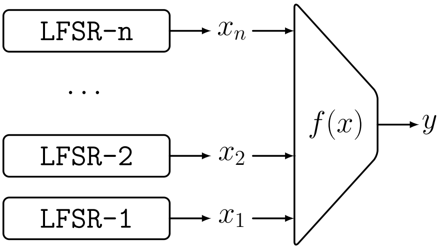
RUHR UNIVERSITÄT BOCHUM RUB

Gefördert durch DFG Deutsche Forschungsgemeinschaft

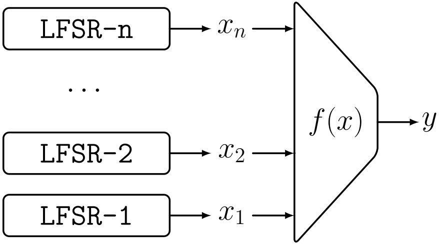
Horst Görtz Institut

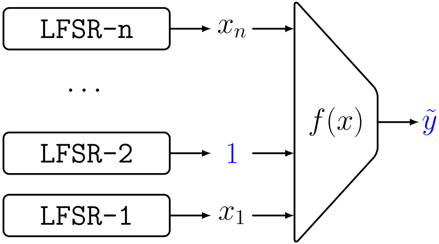
Preliminaries

Resilient Functions



Resilient Functions





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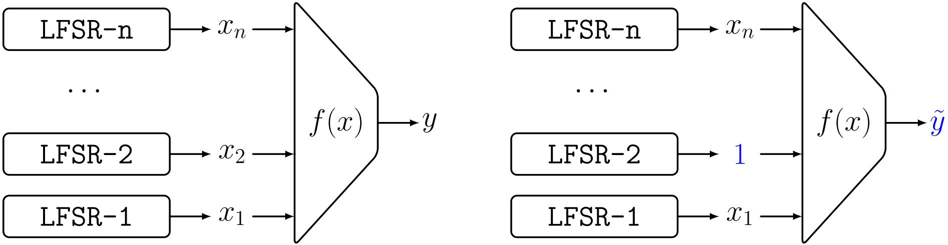

Definition 1: Resilient Boolean Function [Sie84]

Let f to be a **balanced** Boolean function:

$$f \text{ is } t\text{-resilient} \quad \Leftrightarrow \quad \widehat{f}(u) = 0 \;\; \forall u \text{ with } \mathrm{hw}(u) \leq t\,.$$

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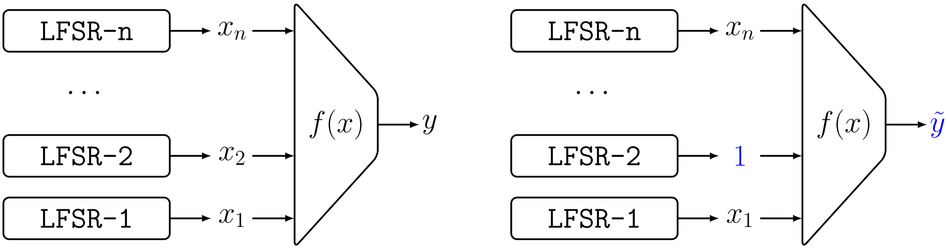

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Walsh Transform: $\widehat{f}(u) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus \langle x, u \rangle}$

Equivalence Relation

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Definition 2: Extended Variable-Permutation Equivalence

Let f and g to be two Boolean functions:

$$f \sim g \qquad \Leftrightarrow \qquad \forall x \in \mathbb{F}_2^n, \quad g(x) = f \circ P(x \oplus a) \oplus b$$

with P, a mapping corresponding to a permutation of n variables, and $a \in \mathbb{F}_2^n$, $b \in \mathbb{F}_2$.

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Definition 3: Representative Function

For each class of equivalence relation, it is **the lexicographically smallest** one.

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Lemma 1:

$$f \text{ is } t\text{-resilient and } f \sim g \quad \Rightarrow \quad g \text{ is } t\text{-resilient}$$

Algebraic Degree

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Lemma 2 [Sie84]:

$$f \text{ is } t\text{-resilient} \;\Rightarrow\; \deg(f) = \begin{cases} n - t - 1 & \text{if } t < n - 1\,, \\ 1 & \text{if } t = n - 1\,. \end{cases}$$

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$(n-1)$-Resilient Function:

$$f(x_1, \ldots, x_n) = x_1 \oplus \ldots \oplus x_n$$

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$(n-2)$-Resilient Function:

$$f(x_1, \ldots, x_n) = x_2 \oplus \ldots \oplus x_n$$

Siegenthaler's Construction [Sie84]

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Theorem 1

Let f to be a balanced Boolean function with $n+1$ variables:

$$f(x, x_{n+1}) = \overline{x_{n+1}} \cdot f_0(x) \oplus x_{n+1} \cdot f_1(x) \quad \forall\, x \in \mathbb{F}_2^n \text{ and } x_{n+1} \in \mathbb{F}_2 \,.$$

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f is a $(t+1)$-resilient if and only if

- *both f_0 and f_1 are t-resilient functions, and*
- *for any $\alpha \in \mathbb{F}_2^n$ with $\mathrm{hw}(\alpha) = t+1$, $\widehat{f_1}(\alpha) = -\widehat{f_0}(\alpha)$.*

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Definition 4: Type-1 Extension

$$f_1(x) = f_0(x) \oplus 1 \quad \Rightarrow \quad f(x, x_{n+1}) = f_0(x) \oplus x_{n+1}$$

$(n-3)$-Resilient Functions [CCCS91]

- $f(x_1, \ldots, x_n) = x_3 \oplus \ldots \oplus x_n \qquad (n \geq 3)\,,$
- $f(x_1, \ldots, x_n) = x_1x_2 \oplus x_3 \oplus \ldots \oplus x_n \qquad (n \geq 3)\,,$
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- $f(x_1, \ldots, x_n) = x_1x_2 \oplus x_1x_3 \oplus x_2x_4 \oplus x_3x_4 \oplus x_3 \oplus \ldots \oplus x_n \qquad (n \geq 4)\,.$

$(n-4)$-Resilient Functions

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Theorem 2 (Tarannikov and Kirienko [TK00])

Any $(n-m)$-resilient n-variable representative function is in the form of

$$g(x_1, \ldots, x_q) \oplus x_{q+1} \oplus \ldots \oplus x_n \quad \textit{with} \quad q \leq p(m).$$

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Example

$$p(3) = 4$$

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Tarannikov & Kirienko [TK00]

$$p(4) = 10$$

An Algorithm for Classifying $(n-m)$-Resilient Functions

Basic Approach

Siegenthaler's Construction

$$f(x, x_{n+1}) = \overline{x_{n+1}} \cdot f_0(x) \oplus x_{n+1} \cdot f_1(x) \quad \forall\, x \in \mathbb{F}_2^n \text{ and } x_{n+1} \in \mathbb{F}_2 \,.$$

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- $\mathcal{R}^*_{n,t}$: the set of **representative** t-resilient n-variable functions

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Computational Complexity of Building $\mathcal{R}^\dagger_{n+1,t+1}$: $\quad |\mathcal{R}_{n,t}|^2$

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$$f \text{ is representative} \quad \Rightarrow \quad f_0 \text{ is representative}\,.$$

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$\Rightarrow$ Comp. Comp.: $|\mathcal{R}^*_{n,t}| \cdot |\mathcal{R}^*_{n,t}| \cdot 2^{n+1} \cdot n!$

Technique 1

Lemma 4:

In Siegenthaler's construction:

$$f_0 \text{ is a type-1 extension} \quad \Rightarrow \quad f \notin \mathcal{R}^{\dagger}_{n+1,t+1}$$

i.e., f cannot be a non-type-1 representative resilient function.

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Lemma 4:

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$\Rightarrow$ Comp. Comp.: $\left(|\mathcal{R}^{\dagger}_{n,t}| \cdot |\mathcal{R}^{*}_{n,t}|\right) \cdot \left(2^{n+1} \cdot n!\right)$

Technique 3

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Lemma 6:

In Siegenthaler's construction,
the two functions f_0 and f_1 from $\mathcal{R}_{n,t}$ can form a function in $\mathcal{R}_{n+1,t+1}$, if

$$\{|\widehat{f_0^*}(\alpha)| \mid \alpha \in \mathbb{F}_2^n, \mathrm{hw}(\alpha) = t+1\} = \{|\widehat{f_1^*}(\alpha)| \mid \alpha \in \mathbb{F}_2^n, \mathrm{hw}(\alpha) = t+1\}$$

Results on the Number of Representative Pairs

Number of representative pairs to be considered for building $\mathcal{R}^{\dagger}_{n,n-4}$

n	5	6	7	8	9	10	11
N_0	1 711	32 896	167 331	259 560	284 635	289 180	289 941
N_1	1 429	26 385	89 855	43 874	8 009	773	62
N_2	1 266	24 356	79 631	28 450	1 919	61	3
N_3	133	1 911	6 423	1 779	149	8	1

Computations for Each Representative Pair

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For each representative pair (f_0^*, f_1^*), we need to consider all equivalent functions to f_1^*.

$$f_1(x) = f_1^* \circ P(x \oplus a) \oplus b$$

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Since $\widehat{f_1}(\alpha) = (-1)^{\langle a,\alpha\rangle \oplus b} \cdot \widehat{f_1^*}(P(\alpha))$,

$$|\widehat{f_1^*}(P(\alpha))| = |\widehat{f_0^*}(\alpha)|$$

Summary

- Classification of all t-resilient functions with $(t+4)$ variables up-to the extended variable-permutation equivalence
- There are only 761 of such functions.

n	4	5	6	7	8	9	10
$\lvert\mathcal{R}^*_{n,n-4}\rvert$	58	256	578	720	754	760	761

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Thank you for your attention!